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An explicit expression of the dark N-soliton solution of the MKdV equation by means of the Darboux transformation

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Abstract. A Darboux transformation is developed for generating dark multi-soliton solutions of the MKdV equation. The Darboux matrices are found explicitly in recursive manner and a system of linear algebraic equations is derived for determining the dark N-soliton solution. By means of the Binet–Cauchy formula an explicit expression of the dark N-soliton solution is obtained.

Although the inverse scattering transform is the most systematic method for giving soliton solutions of certain nonlinear evolution equations (Gardner *et al* 1967, Zakharov and Shabat 1971, Ablowitz *et al* 1973), the Darboux transform has its special meaning (Levi *et al* 1981, Asano and Kato 1981). It is more simple and it can generate multi-soliton solutions by a pure algebraic process, when the Darboux matrices are found explicitly in a recursive manner (Chen *et al* 1988, Chen and Huang 1989, Huang 1992).

The inverse scattering transform is also used for finding dark soliton solutions of certain nonlinear evolution equations (Zakharov and Shabat 1973, Kawata and Inoue 1977, 1978). However, it is more involved in these cases. To extend the Darboux transformation to generate the dark soliton solutions is desirable (Asano and Kato 1981, 1984). Recently, the Darboux transformation has been examined for generating the dark soliton solutions of the MKdV equation (Chau *et al* 1991). Unfortunately, the calculation procedure in this case is too complicated and cannot be used in practice.

The same problem is re-examined in the present paper. For this purpose we developed a Darboux transformation which has the same form as those for bright soliton solutions of the NLS equation (Chen *et al* 1988). The Darboux matrices are found explicitly in a recursive manner and then a system of linear algebraic equations for giving the dark N-soliton solution is derived. By using the Binet–Cauchy formula, an explicit expression of the dark N-soliton solution is obtained by a similar procedure as that used in the case of the bright soliton (Chen *et al* 1989).

The MKdV equation

$$u_t + u_{xxx} - 6u^2 u_x = 0 \tag{1}$$

is known not to have bright soliton solutions. The Lax equations are

$$\partial_x F(\lambda) = L(\lambda) F(\lambda) \tag{2}$$

$$\partial_t F(\lambda) = M(\lambda) F(\lambda) \tag{3}$$

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where

$$L(\lambda) = -\mathrm{i}\lambda\sigma_3 + U \tag{4}$$

$$M(\lambda) = -\mathrm{i}4\lambda^3\sigma_3 + 4\lambda^2 U - \mathrm{i}2\lambda(U^2 + U_x)\sigma_3 - U_{xx} + 2U^3 \tag{5}$$

$$U = \begin{pmatrix} 0 & u \\ u & 0 \end{pmatrix} \qquad \text{or} \qquad U = -\begin{pmatrix} 0 & u \\ u & 0 \end{pmatrix}. \tag{6}$$

Now we consider dark soliton solutions of (1) that have finite values at spatial infinities

$$u \to c \qquad \text{as } x \to \pm\infty \tag{7}$$

where c is assumed to be positive.

The asymptotic behaviours of Jost solutions of the Lax equations in the limit as $|\lambda| \to \infty$ are independent of particular solutions of u. Therefore, as in the paper of Chen *et al* (1988), the Jost solution $F_n(\lambda)$ can be defined recursively

$$F_n(\lambda) = D_n(\lambda)F_{n-1}(\lambda) \tag{8}$$

where

$$D_n(\lambda) = I + \frac{\lambda_n - \mu_n}{\lambda - \lambda_n} P_n \tag{9}$$

$$\mu_n = -\lambda_n = \text{real} \tag{10}$$

and P_n is a 2×2 matrix independent of λ. A minor change here is that λ_n is real, since (2) can be transformed into an Hermitian eigenvalue equation.

As in the paper of Chen *et al* (1988), we have

$$U_n = U_{n-1} - \mathrm{i}(\lambda_n - \mu_n)[P_n, \sigma_3] \tag{11}$$

$$P_n = \frac{F_{n-1}(\mu_n)\begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix}(\gamma_n \delta_n)F_{n-1}^{-1}(\lambda_n)}{(\gamma_n \delta_n)F_{n-1}^{-1}(\lambda_n)F_{n-1}(\mu_n)\begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix}} \tag{12}$$

where $\alpha_n, \ldots$ are constants. P_n is clearly a projection

$$P_n^2 = P_n. \tag{13}$$

In the case of bright solitons, $(P_n)_{12}$ and $(P_n)_{21}$ vanish at spatial infinities and then u_n also vanishes. In the case of dark solitons (7), $(P_n)_{12}$ and $(P_n)_{21}$ are shown not to vanish in these limits. One way is to assume

$$U_n = (-1)^n \begin{pmatrix} 0 & u_n \\ u_n & 0 \end{pmatrix}. \tag{14}$$

Therefore, from (11), we have

$$(-1)^n u_n = (-1)^{n-1} u_{n-1} + \mathrm{i}4\lambda_n (P_n)_{12} \tag{15}$$

$$(-1)^n u_n = (-1)^{n-1} u_{n-1} - \mathrm{i}4\lambda_n (P_n)_{21}. \tag{16}$$

It is easily seen that P_n must satisfy three conditions

$$(P_n)_{12} = -(P_n)_{21} \tag{17}$$

$$(P_n)_{21} = \text{pure imaginary} \tag{18}$$

and

$$(P_n)_{21} \to (-1)^n \frac{\mathrm{i}c}{2\lambda_n} \qquad \text{as } x \to \pm\infty \tag{19}$$

on account of (16).

With the condition (7), a simple solution of (1) is clearly

$$u_0 = c. \tag{20}$$

The corresponding Jost solution $F_0(\lambda)$ is

$$F_0(\lambda) = \frac{1}{\sqrt{1+\rho^2}} \begin{pmatrix} \mathrm{e}^{-\theta} & -\rho\, \mathrm{e}^{\theta} \\ \rho\, \mathrm{e}^{-\theta} & \mathrm{e}^{\theta} \end{pmatrix} \tag{21}$$

where

$$\theta = \sqrt{c^2 - \lambda^2}\,[x + (4\lambda^2 + 2c^2)t] \tag{22}$$

$$\rho = (-\sqrt{c^2 - \lambda^2} + \mathrm{i}\lambda)/c. \tag{23}$$

It can be shown that the three conditions (17)-(19) are satisfied by choosing

$$\alpha_n = \exp[(c^2 - \lambda_n^2)^{1/2} x_n] \qquad \beta_n = -\mathrm{i}\exp[(c^2 - \lambda_n^2)^{-1/2} x_n] \tag{24}$$

$$\gamma_n = \exp[(c^2 - \lambda_n^2)^{-1/2} x_n] \qquad \delta_n = \mathrm{i}\exp[(c^2 - \lambda_n^2)^{1/2} x_n] \tag{25}$$

where x_n is a real constant. We have also

$$F_0^{-1}(-\lambda) = \frac{1}{\sqrt{1+\rho^{-2}}} \begin{pmatrix} \mathrm{e}^{\theta} & \rho^{-1}\mathrm{e}^{\theta} \\ -\rho^{-1}\mathrm{e}^{-\theta} & \mathrm{e}^{-\theta} \end{pmatrix}. \tag{26}$$

From them we have

$$F_0(-\lambda_n)\begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix}(\gamma_n \delta_n) F_0^{-1}(\lambda_n)$$

$$= \begin{pmatrix} 2 - \mathrm{i}\bar{\rho}_n \mathrm{e}^{-2\theta_n} + \mathrm{i}\rho_n \mathrm{e}^{2\theta_n} & \bar{\rho}_n - \rho_n + \mathrm{i}(\mathrm{e}^{-2\theta_n} + \mathrm{e}^{2\theta_n}) \\ -\bar{\rho}_n + \rho_n - \mathrm{i}(\mathrm{e}^{-2\theta_n} + \mathrm{e}^{2\theta_n}) & 2 + \mathrm{i}\rho_n \mathrm{e}^{-2\theta_n} - \mathrm{i}\bar{\rho}_n \mathrm{e}^{2\theta_n} \end{pmatrix} \tag{27}$$

where

$$\theta_n = \sqrt{c^2 - \lambda_n^2}\,[x - x_n + (4\lambda_n^2 + 2c^2)t] \tag{28}$$

$$\rho_n = (-\sqrt{c^2 - \lambda_n^2} - \mathrm{i}\lambda_n)/c. \tag{29}$$

We thus see that

$$\bar{\rho}_n - \rho_n = \mathrm{i}2\lambda_n/c. \tag{30}$$

When $n = 1$, from (27) we can see that the conditions (17) and (18) are satisfied. In the limit as $x \to \infty$, we have

$$\begin{pmatrix} \mathrm{i}\rho_1 \mathrm{e}^{2\theta_1} & \mathrm{i}\,\mathrm{e}^{2\theta_1} \\ -\mathrm{i}\,\mathrm{e}^{2\theta_1} & -\mathrm{i}\bar{\rho}_1 \mathrm{e}^{2\theta_1} \end{pmatrix}. \tag{31}$$

It is clear that the condition (19) is satisfied when $n = 1$. We can show that the three conditions (17)-(19) are satisfied in recursive manner.

From these formulae, the dark 1-soliton solution of the MKdV equation is

$$u_1 = -c + i4\lambda_1(P_1)_{21} = c - 2\frac{c^2-\lambda_1^2}{c}\left(1+\frac{\lambda_1}{c}\cosh[2\theta_1]\right)^{-1} \tag{32}$$

where $-c < \lambda_1 < c$.

Though the dark multi-soliton solutions of the MKdV equation can be obtained in a recursive manner, the calculation processes are still involved. As in the paper of Chen *et al* (1988), we can derive a system of linear algebraic equations for determining directly the N-soliton solution. From (8), we have

$$F_N(\lambda) = G_N(\lambda)F_0(\lambda) \tag{33}$$

where

$$G_N(\lambda) = D_N(\lambda), \ldots, D_1(\lambda). \tag{34}$$

We have also

$$F_N^{-1}(\lambda) = F_0^{-1}(\lambda)G_N^{-1}(\lambda) \tag{35}$$

$$G_N^{-1}(\lambda) = D_1^{-1}(\lambda), \ldots, D_N^{-1}(\lambda) \tag{36}$$

$$D_n^{-1}(\lambda) = I + \frac{\mu_n - \lambda_n}{\lambda - \mu_n} P_n. \tag{37}$$

Similar to that in the paper of Chen *et al* (1988), we have

$$G_N(\lambda) = I + \sum_{n=1}^{N} \frac{1}{\lambda-\lambda_n} a_n^{-1}\sigma_2 G_N^{-1}(\lambda_n)^{\mathrm{T}}\sigma_2 \tag{38}$$

where the superscript T means the transpose, and

$$a_n = \prod_{m\neq n} \frac{\lambda_n - \lambda_m}{\lambda_n - \mu_m} \frac{1}{\lambda_n - \mu_n}. \tag{39}$$

From (21), we have

$$\sigma_2 F_0^{-1}(\lambda)^{\mathrm{T}}\sigma_2 = F_0(\lambda). \tag{40}$$

Equation (38) can be rewritten as

$$F_N(\lambda)F_0^{-1}(\lambda) = I + \sum_{n=1}^{N}\frac{1}{\lambda-\lambda_n} a_n^{-1}\sigma_2 F_N^{-1}(\lambda_n)^{\mathrm{T}}\sigma_2 F_0^{-1}(\lambda_n). \tag{41}$$

Owing to (17) and (18), the projection matrix P_n must satisfy

$$\overline{\sigma_2 P_n^{\mathrm{T}}\sigma_2} = P_n. \tag{42}$$

We then have

$$\overline{\sigma_2 D_n^{-1}(\lambda_m)^{\mathrm{T}}\sigma_2} = D_n(-\lambda_m) \tag{43}$$

and

$$F_0(-\lambda_m) = \overline{F_0(\lambda_m)} = \overline{\sigma_2 F_0^{-1}(\lambda_m)^{\mathrm{T}}\sigma_2}. \tag{44}$$

Therefore, we have

$$F_N(-\lambda_m) = \overline{\sigma_2 F_N^{-1}(\lambda_m)^{\mathrm{T}}\sigma_2}. \tag{45}$$

Setting $\lambda=\mu_m=-\lambda_m$ in (41), we obtain

$$\overline{\sigma_2 F_N^{-1}(\lambda_m)^{\mathrm{T}}\sigma_2}\;\overline{F_0^{-1}(\lambda_m)}=I-\sum_{n=1}^{N}\frac{1}{\lambda_m+\lambda_n}a_n^{-1}\sigma_2 F_N^{-1}(\lambda_n)^{\mathrm{T}}\sigma_2 F_0^{-1}(\lambda_n). \tag{46}$$

With the same procedure as that in the paper by Chen *et al* (1988), we can show that

$$\sigma_2 F_N^{-1}(\lambda_n)^{\mathrm{T}}\sigma_2=\psi(\lambda_n)\begin{pmatrix}\frac{\gamma_n}{\delta_n} & 1\end{pmatrix} \tag{47}$$

where $\psi(\lambda)$ is the 2×1 Jost solution in the case of $u=u_N$. We write

$$(\gamma_n\delta_n)F_0^{-1}(\lambda_n)=\frac{1}{\sqrt{1+\rho_n^2}}(h_n k_n) \tag{48}$$

where

$$h_n=\mathrm{e}^{\theta_n}+\mathrm{i}\rho_n\,\mathrm{e}^{-\theta_n} \tag{49}$$

$$k_n=\rho_n\,\mathrm{e}^{\theta_n}-\mathrm{i}\,\mathrm{e}^{-\theta_n}. \tag{50}$$

Substituting these formulae into (46), the 21 and 22 elements of it are

$$\overline{\left(\frac{\psi_2(\lambda_m)}{\delta_m\sqrt{1+\rho_m^2}}\right)}(\bar{h}_m\quad \bar{k}_m)=(0\quad 1)-\sum_{n=1}^{N}\frac{1}{\lambda_m+\lambda_n}\left(\frac{\psi_2(\lambda_n)}{\delta_n\sqrt{1+\rho_n^2}}\right)(h_n\quad k_n). \tag{51}$$

Introducing symbols

$$\Psi_{2n}=a_n^{-1}\frac{\psi_2(\lambda_n)}{\delta_n\sqrt{1+\rho_n^2}} \tag{52}$$

$$A_{nm}=h_n\frac{1}{\lambda_n+\lambda_m}a_m^{-1}\bar{h}_m^{-1} \tag{53}$$

$$\bar{B}_{nm}=k_n\frac{1}{\lambda_n+\lambda_m}a_m^{-1}\bar{k}_m^{-1} \tag{54}$$

$$\bar{K}_m=a_m^{-1}\bar{k}_m^{-1} \tag{55}$$

equation (46) can be rewritten in the matrix form

$$\bar{\Psi}_2=-\Psi_2 A \tag{56}$$

$$\Psi_2=K-\bar{\Psi}_2 B. \tag{57}$$

The dark N-soliton solution of the MKdV equation can be expressed as

$$(-1)^N u_N=c-\mathrm{i}2(R_N)_{21} \tag{58}$$

where

$$(R_N)_{21}=\sum_{n=1}^{N}a_n^{-1}\left(\frac{\psi_2(\lambda_n)}{\delta_n\sqrt{1+\rho_n^2}}\right)h_n=\Psi_2 H^{\mathrm{T}} \tag{59}$$

$$H_n=h_n. \tag{60}$$

From (56), (57) and (59), we obtain

$$(R_N)_{21}=K(I-AB)^{-1}H^{\mathrm{T}}=\frac{\det(I+Q')}{\det(I+Q)}-1 \tag{61}$$

where

$$Q = -AB \tag{62}$$

$$Q' = Q + H^{\mathrm{T}}K. \tag{63}$$

By using the known formulae, we have

$$\det(I+Q) = 1 + \sum_{r=1}^{N} \sum_{1 \leqslant n_1 < n_2 < \ldots < n_r \leqslant N} Q(n_1, n_2, \ldots, n_r) \tag{64}$$

$$Q(n_1, n_2, \ldots, n_r) = (-1)^r \sum_{1 \leqslant m_1 < m_2 < \ldots < m_r \leqslant N} A(n_1, n_2, \ldots, n_r; m_1, m_2, \ldots, m_r) \times B(m_1, m_2, \ldots, m_r; n_1, n_2, \ldots, n_r) \tag{65}$$

where $A(n_1, n_2, \ldots, n_r; m_1, m_2, \ldots, m_r)$ is a minor of order r of A which is a determinant of a submatrix of A by remaining $(n_1, n_2, \ldots, n_r)$th rows and $(m_1, m_2, \ldots, m_r)$th columns. $Q(n_1, n_2, \ldots, n_r)$ means a principal minor. From the particular forms of A and B, we find

$$A(n_1, n_2, \ldots, n_r; m_1, m_2, \ldots, m_r)B(m_1, m_2, \ldots, m_r; n_1, n_2, \ldots, n_r) = \prod_n h_n(a_n k_n)^{-1} \prod_m \bar{k}_m(a_m \bar{h}_m)^{-1} \prod_{n<n'} (\lambda_n - \lambda_{n'})^2 \times \prod_{m<m'} (\lambda_m - \lambda_{m'})^2 \prod_{n,m} (\lambda_n + \lambda_m)^{-2} \tag{66}$$

where

$$n, n' \in \{n_1, n_2, \ldots, n_r\} \tag{67a}$$

$$m, m' \in \{m_1, m_2, \ldots, m_r\}. \tag{67b}$$

Q' can be rewritten as

$$Q' = -A'B' \tag{68}$$

where A' is an $N \times (N+1)$ matrix such that

$$A'_{n0} = \mathrm{i}H_n \qquad A'_{nm} = A_{nm} \qquad n, m = 1, 2, \ldots, N \tag{69}$$

and B' is an $(N+1) \times N$ matrix

$$B'_{0n} = \mathrm{i}K_n \qquad B'_{mn} = B_{mn} \qquad n, m = 1, 2, \ldots, N. \tag{70}$$

We then have

$$Q'(n_1, n_2, \ldots, n_r) = (-1)^r \sum_{0 \leqslant m_1 < m_2 < \ldots < m_r \leqslant N} A'(n_1, n_2, \ldots, n_r; m_1, m_2, \ldots, m_r) \times B'(m_1, m_2, \ldots, m_r; n_1, n_2, \ldots, n_r). \tag{71}$$

The summation is clearly decomposed of two: one is extended $m_1 = 0$, the other $m_1 \geqslant 1$. The latter is just $Q(n_1, n_2, \ldots, n_r)$. Therefore, we have

$$\det(I+Q') - \det(I+Q) = \sum_{r=1}^{N} (-1)^r \sum_{1 \leqslant n_1 < n_2 < \ldots < n_r \leqslant N} \sum_{1 \leqslant m_2 < \ldots < m_r \leqslant N} \times A'(n_1, n_2, \ldots, n_r; 0, m_2, \ldots, m_r) \times B'(0, m_2, \ldots, m_r; n_1, n_2, \ldots, n_r). \tag{72}$$

Similar to (66), we have

$$A'(n_1, n_2, \ldots, n_r; 0, m_2, \ldots, m_r)B'(0, m_2, \ldots, m_r; n_1, n_2, \ldots, n_r)$$
$$= (-1)\prod_n h_n(a_n k_n)^{-1} \prod_m \bar{k}_m(a_m \bar{h}_m)^{-1} \prod_{n<n'} (\lambda_n - \lambda_{n'})^2$$
$$\times \prod_{m<m'} (\lambda_m - \lambda_{m'})^2 \prod_{n,m} (\lambda_n + \lambda_m)^{-2} \tag{73}$$

where

$$n, n' \in \{n_1, n_2, \ldots, n_r\} \tag{74a}$$

$$m, m' \in \{m_2, \ldots, m_r\}. \tag{74b}$$

Though (73) is similar to (66) in form, (73) is essentially different from (66) on account of the difference between (74) and (67).

Therefore, we obtain

$$(R_N)_{21} = E_N / D_N \tag{75}$$

where

$$D_N = \sum_{r=0}^{N} (-1)^r \sum_{1\le n_1<n_2<\ldots<n_r\le N} \; \sum_{1\le m_1<m_2<\ldots<m_r\le N}$$
$$\times D_N(n_1, n_2, \ldots, n_r; m_1, m_2, \ldots, m_r) \tag{76}$$

$$D_N(n_1, n_2, \ldots, n_r; m_1, m_2, \ldots, m_r)$$
$$= \prod_n a_n^{-1} h_n \prod_m a_m^{-1} \bar{k}_m \prod_{n<n'} (\lambda_n - \lambda_{n'})^2 \prod_{m<m'} (\lambda_m - \lambda_{m'})^2$$
$$\times \prod_{n,m} (\lambda_n + \lambda_m)^{-2} \prod_{\tilde{n}} k_{\tilde{n}} \prod_{\tilde{m}} \bar{h}_{\tilde{m}} \tag{77}$$

with the condition (67) and

$$\tilde{n} \notin \{n_1, n_2, \ldots, n_r\} \qquad \tilde{m} \notin \{m_1, m_2, \ldots, m_r\} \tag{78}$$

as well as

$$E_N = \sum_{r=1}^{N} (-1)^{r+1} \sum_{1\le n_1<n_2<\ldots<n_r\le N} \; \sum_{1\le m_2<\ldots<m_r\le N}$$
$$\times E_N(n_1, n_2, \ldots, n_r; 0, m_2, \ldots, m_r) \tag{79}$$

$$E_N(n_1, n_2, \ldots, n_r; 0, m_2, \ldots, m_r)$$
$$= \prod_n a_n^{-1} h_n \prod_m a_m^{-1} \bar{k}_m \prod_{n<n'} (\lambda_n - \lambda_{n'})^2 \prod_{m<m'} (\lambda_m - \lambda_{m'})^2$$
$$\times \prod_{n,m} (\lambda_n + \lambda_m)^{-2} \prod_{\tilde{n}} k_{\tilde{n}} \prod_{\tilde{m}} \bar{h}_{\tilde{m}} \tag{80}$$

with the condition (74) and

$$\tilde{n} \notin \{n_1, n_2, \ldots, n_r\} \qquad \tilde{m} \notin \{m_2, \ldots, m_r\}. \tag{81}$$

From (58) and (75), we obtain an explicit expression of the dark N-soliton solution of the MKdV equation. Up to now it has never been found by all means.

When $N=1$, from (58) and (75) we obtain (32) also. When $N=2$, we have

$$D_2=k_1k_2\bar{h}_1\bar{h}_2-a_1^{-1}h_1a_1^{-1}\bar{k}_1(2\lambda_1)^{-2}k_2\bar{h}_2-a_1^{-1}h_1a_2^{-1}k_2(\lambda_1+\lambda_2)^{-2}k_2\bar{h}_1$$
$$-a_2^{-1}h_2a_1^{-1}\bar{k}_1(\lambda_2+\lambda_1)^{-2}k_1\bar{h}_2-a_2^{-1}h_2a_2^{-1}\bar{k}_2(2\lambda_2)^{-2}k_1\bar{h}_1$$
$$+a_1^{-2}a_2^{-2}h_1h_2\bar{k}_1\bar{k}_2(\lambda_1-\lambda_2)^4(2\lambda_1)^{-2}(\lambda_1+\lambda_2)^{-4}(2\lambda_2)^{-2} \quad (82)$$

$$E_2=a_1^{-1}h_1k_2\bar{h}_1\bar{h}_2+a_2^{-1}h_2k_1\bar{h}_1\bar{h}_2$$
$$+a_1^{-2}h_1\bar{k}_1a_2^{-1}h_2\bar{h}_2(\lambda_1-\lambda_2)^2(2\lambda_1)^{-2}(\lambda_2+\lambda_1)^{-2}$$
$$+a_1^{-1}h_1a_2^{-2}h_2\bar{k}_2\bar{h}_1(\lambda_1-\lambda_2)^2(\lambda_1+\lambda_2)^{-2}(2\lambda_2)^{-2}. \quad (83)$$

Substituting (39), (49) and (50), we obtain

$$D_2=\left(\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}\right)^2 4\lambda_1\lambda_2D_2' \quad (84)$$

$$E_2=\left(\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}\right)^2 4\lambda_1\lambda_2E_2' \quad (85)$$

where

$$D_2'=\frac{4}{\lambda_1\lambda_2}\left(1+\frac{\lambda_1}{c}\cosh[2\theta_1]\right)\left(1+\frac{\lambda_2}{c}\cosh[2\theta_2]\right)$$
$$-\frac{1}{(\lambda_1+\lambda_2)^2}\left\{\left[1+\frac{\lambda_1\lambda_2}{c^2}+\frac{\sqrt{c^2-\lambda_1^2}\sqrt{c^2-\lambda_2^2}}{c^2}\right]\cosh^2[\theta_1-\theta_2]\right.$$
$$+\left[1+\frac{\lambda_1\lambda_2}{c^2}-\frac{\sqrt{c^2-\lambda_1^2}\sqrt{c^2-\lambda_2^2}}{c^2}\right]\cosh^2[\theta_1+\theta_2]$$
$$\left.-2\frac{\lambda_1\lambda_2}{c^2}\cosh[\theta_1+\theta_2]\cosh[\theta_1-\theta_2]\right\} \quad (86)$$

$$E_2'=-\mathrm{i}\frac{4}{\lambda_2}\left(\frac{\lambda_1}{c}+\cosh[2\theta_1]\right)\left(1+\frac{\lambda_2}{c}\cosh 2\theta_2\right)$$
$$-\mathrm{i}\frac{4}{\lambda_1}\left(\frac{\lambda_2}{c}+\cosh[2\theta_2]\right)\left(1+\frac{\lambda_1}{c}\cosh[2\theta_1]\right)$$
$$+\mathrm{i}8c^{-1}\{\cosh^2[\theta_1+\theta_2]+\cosh^2[\theta_1-\theta_2]\}$$
$$-\mathrm{i}\frac{1}{\lambda_1+\lambda_2}\left(1+\frac{\lambda_1\lambda_2}{c^2}\right)\{\cosh^2[\theta_1]+\cosh^2[\theta_2]\}. \quad (87)$$

We thus obtain an explicit expression of u_2.

Finally, we ought to show that the Jost solutions obtained by the above procedure satisfy the corresponding Lax equations. From (41) and (47), we have

$$\lim_{\lambda\to\lambda_n}\{\partial_xF_N(\lambda)\}F_N^{-1}(\lambda)$$
$$=\lim_{\lambda\to\lambda_n}\left\{\frac{1}{\lambda-\lambda_n}a_n^{-1}\psi_x(\lambda_n)\begin{pmatrix}\frac{\gamma_n}{\delta_n} & 1\end{pmatrix}\right\}\sigma_2\left\{\psi(\lambda_n)\begin{pmatrix}\frac{\gamma_n}{\delta_n} & 1\end{pmatrix}\right\}^{\mathrm{T}}\sigma_2=0. \quad (88)$$

Similarly, $\{\partial_xF_N(\lambda)\}F_N^{-1}(\lambda)$ vanishes as $\lambda\to\mu_n$. Therefore, λ_n, μ_n, $n=1,2,\ldots,N$ are regular points of $\{\partial_xF_N(\lambda)\}F_N^{-1}(\lambda)$ and, similarly, of $\{\partial_tF_N(\lambda)\}F_N^{-1}(\lambda)$.

From (33) and (35), we have

$$\{\partial_x F_N(\lambda)\}F_N^{-1}(\lambda) = G_{Nx}(\lambda)G_N^{-1}(\lambda) + G_N(\lambda)L_0(\lambda)G_N^{-1}(\lambda) \tag{89}$$

where

$$L_0(\lambda) = -\mathrm{i}\lambda\sigma_3 + C \tag{90}$$

$$C = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}. \tag{91}$$

From the right-hand side of (89) we can see that $\{\partial_x F_N(\lambda)\}F_N^{-1}(\lambda)$ has no cut on the real axis of complex λ-plane. $\{\partial_x F_N(\lambda)\}F_N^{-1}(\lambda)$ is thus analytic everywhere except at $\lambda = \infty$.

We expand $G_N(\lambda)$ into a Taylor series about $\lambda = \infty$:

$$G_N(\lambda) = \sum_{j=0}^{\infty} \alpha_j \lambda^{-j} \tag{92}$$

where

$$\alpha_0 = I \tag{93}$$

$$\alpha_1 = R_N = \tfrac{1}{2}\mathrm{i}(U_N - C). \tag{94}$$

Similarly, we have

$$G_N^{-1}(\lambda) = \sum_{j=0}^{\infty} \beta_j \lambda^{-j} \tag{95}$$

$$\beta_0 = I \qquad \beta_1 = -\alpha_1. \tag{96}$$

Substituting (92) and (95) into (89), we have

$$\{\partial_x F_N(\lambda)\}F_N^{-1}(\lambda) = L_N(\lambda) + \mathrm{O}(|\lambda|^{-1}) \tag{97}$$

as $|\lambda| \to \infty$. Therefore, $\{\partial_x F_N(\lambda)\}F_N^{-1}(\lambda) - L_N(\lambda)$ is analytic in the whole complex λ-plane and tends to zero as $|\lambda| \to \infty$, by Liouville theorem it is equal to zero. This yields

$$G_{Nx}(\lambda)G_N^{-1}(\lambda) + G_N(\lambda)L_0(\lambda)G_N^{-1}(\lambda) = L_N(\lambda) \tag{98}$$

or

$$\partial_x F_N(\lambda) = L_N(\lambda)F_N(\lambda). \tag{99}$$

Similarly, we have

$$\{\partial_t F_N(\lambda)\}F_N^{-1}(\lambda) = G_{Nt}(\lambda)G_N^{-1}(\lambda) + G_N(\lambda)M_0(\lambda)G_N^{-1}(\lambda) \tag{100}$$

where

$$M_0(\lambda) = -\mathrm{i}4\lambda^3\sigma_3 + 4\lambda^2 C - \mathrm{i}2\lambda C^2\sigma_3 + 2C^3. \tag{101}$$

$\{\partial_t F_N(\lambda)\}F_N^{-1}(\lambda)$ is thus analytic everywhere except $\lambda = \infty$. Owing to (101), the right-hand side of (100) is

$$4\lambda^2(-\mathrm{i}\lambda\sigma_3 + U_N) - 4\lambda\alpha_{1x} - \mathrm{i}2\lambda C^2\sigma_3 - 4(\alpha_{2x} + \alpha_{1x}\beta_1)$$
$$-\mathrm{i}2(\alpha_1 C^2\sigma_3 + C^2\sigma_3\beta_1) + 2C^3 + \mathrm{O}(|\lambda|^{-1}) \tag{102}$$

on account of (98). From (99) we have also

$$G_{Nxx}(\lambda) + 2G_{Nx}(\lambda)L_0(\lambda) = (U_N^2 - C^2 + U_{Nx})G_N(\lambda). \tag{103}$$

Multiplying it by $\sigma_3 G_N^{-1}(\lambda)$ and expanding the resultant equation about $\lambda=\infty$ we obtain

$$2(-\mathrm{i})\alpha_{1x}=(U_N^2-C^2+U_{Nx})\sigma_3 \tag{104}$$

$$\alpha_{1xx}\sigma_3+2(-\mathrm{i})(\alpha_{1x}\beta_1+\alpha_{2x})+2\alpha_{1x}C\sigma_3=(U_N^2-C^2+U_{Nx})(\alpha_1\sigma_3+\sigma_3\beta_1). \tag{105}$$

Substituting these terms into (102), (102) is equal to

$$M_N(\lambda)+\mathrm{O}(|\lambda|^{-1}). \tag{106}$$

A similar derivation as for (99) yields

$$\partial_t F_N(\lambda)=M_N(\lambda)F_N(\lambda). \tag{107}$$

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